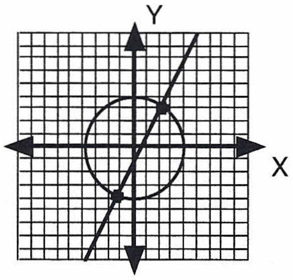
Lesson 27 Solving Systems of Equations, Lines and Conic Sections

In Algebra 1 we learned how to find the solution of two lines that intersect at a single point. We did this first by graphing, then using substitution and elimination. In this lesson we will study ways, using the same techniques, to find the intersection of a line (linear equation with first degree variables) and either a circle, ellipse, parabola, or hyperbola (non-linear equation with second degree variables). Then when this is mastered, we will move to the solutions of two non-linear equations.

The best way to start is to do some examples and talk our way through them. If you get stuck on the first example, you can do examples 2 and 3 first, and then come back to this one.

Figure 1



Example 1

Find the solution of : $\begin{cases} X^2 + Y^2 = 16 \\ Y = 2X - 1 \end{cases}$

1) Identify the nature of the equations. Are they linear or non-linear, and if non-linear, which conic section is represented? The first equation represents a circle and the second equation a line.

2) Sketch a graph of each to give an estimate of the solutions. Use graph paper and a compass for more accuracy.

3) We have two options now, either to square the second equation and then use elimination, or substitute the second into the first. I choose the second option.

$$X^2 + (2X-1)^2 = 16$$
$$X^2 + 4X^2 - 4X + 1 = 16$$
$$5X^2 - 4X - 15 = 0$$

4) First we look to see if it is possible to factor to find the solutions. It isn't, so we use the quadratic formula.

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(-15)}}{2(5)} = \frac{4 \pm \sqrt{16+300}}{10} = \frac{4 \pm 2\sqrt{79}}{10}$$

$$X = \frac{2+\sqrt{79}}{5}, \frac{2-\sqrt{79}}{5}$$

5) Next we put these answers into Y=2X-1 to find the solutions for Y.

$$Y = 2\left(\frac{2+\sqrt{79}}{5}\right) - 1 = \frac{4}{5} + \frac{2\sqrt{79}}{5} - \frac{5}{5} = \frac{2\sqrt{79}}{5} - \frac{1}{5} = \frac{2\sqrt{79}-1}{5}$$

$$Y = 2\left(\frac{2-\sqrt{79}}{5}\right) - 1 = \frac{4}{5} - \frac{2\sqrt{79}}{5} - \frac{5}{5} = -\frac{1}{5} - \frac{2\sqrt{79}}{5} = \frac{-1-2\sqrt{79}}{5}$$

6) The final solutions are: (Approximate $\sqrt{79}$ to be 9, then put this into the solutions to see how close it is to the graph).

$$\left(\frac{2+\sqrt{79}}{5}, \frac{2\sqrt{79}-1}{5}\right) \& \left(\frac{2-\sqrt{79}}{5}, \frac{-1-2\sqrt{79}}{5}\right)$$

$$\left(\frac{2+9}{5}, \frac{2\cdot 9-1}{5}\right) \& \left(\frac{2-9}{5}, \frac{-1-2\cdot 9}{5}\right) - \left(\frac{11}{5}, \frac{17}{5}\right) \& \left(-\frac{7}{5}, -\frac{19}{5}\right)$$

Plot these points on the graph, and they fit!

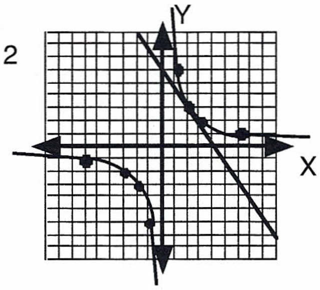
Note that in some cases the graphs of two equations will not intersect at all. In these cases, there are no real solutions that satisfy both equations. In the case of a line and a circle, they may intersect in 2 places, in only 1 place, or not at all. Other combinations of linear equations and/or conic sections have other possibilities.

Let's back up to gain perspective on this problem. First we substituted Y=2X-1 into the first equation so that we only had one variable to to solve for, in this case X. Since we couldn't factor the equation, we used the quadratic formula to find the two values of X. We knew by our graph that there were two points where the graphs intersected, so we expected two solutions. So, after we found the two values for X, we had to replace them in either of the equations to find the corresponding values of Y. We chose the easier of the two and, as a result, had the final solution. Then, to make sure that our solution was the same as the coordinates on the graph, we substituted 9 for the approximate value of the square root of 79. And, sure enough, this is where we expected to find the points.

If this seems hard, relax, the problems are not usually as hard as this one. Let's do another example, and an easier one.

Example 2 Find the solution of : $\begin{cases} XY = 6 \\ 3X + 2Y = 12 \end{cases}$

Figure 2



$$2Y = -3X + 12$$

$$Y = \frac{-3X}{2} + 6$$

1) Identify the nature of the equations. The first equation represents a hyperbola and the second equation a line.

2) Sketch a graph of each to give an estimate of the solutions. Use graph paper and a compass for more accuracy.

3) Substitute the second into the first.

$$X(-3/2X + 6) = 6$$

$$-3/2X^2 + 6X = 6$$

$$-3X^2 + 12X - 12 = 0$$

$$X^2 - 4X + 4 = 0$$

4) First we look to see if it is possible to factor to find the solutions. It is, and and the factor is X=2.

$$X^2 - 4X + 4 = 0$$

$$(X-2)(X-2) = 0$$

5) Next we put this answer into XY=6 to find the solution for Y.

$$(2)Y = 6$$

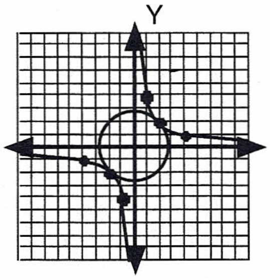
$$Y = 3$$

6) The final solution is: (2,3) which agrees with the graph.

Example 3 Find the solution of :

$$\begin{cases} XY = 4 \\ X^2 + Y^2 = 8 \end{cases}$$

Figure 3



1) The first equation is a hyperbola and the second, a circle

2) Sketch a graph of each to give an estimate of the solutions.

3) Substitute the first into the second to solve for one variable.

$$(4/Y)^2 + Y^2 = 8$$

$$16/Y^2 + Y^2 = 8$$

$$16 + Y^4 = 8Y^2$$

$$Y^4 - 8Y^2 + 16 = 0$$

4) Factor to find the solutions.

$$(Y^2 - 4)^2 = 0$$

$$(Y^2 - 4)(Y^2 - 4) = 0$$

$$(Y+2)(Y-2)(Y+2)(Y-2) = 0$$

5) Put Y=±2, into XY=4 to find the solution for X.

$(2)X = 4$ $\quad$ $(-2)X = 4$

$X = 2$ $\quad$ $X = -2$

6) The final solutions are: (2,2) and (-2,-2) which agree with the graph.

Practice Problems

1) $\begin{cases} Y = X+3 \\ X^2+Y^2=9 \end{cases}$

3) $\begin{cases} Y = X^2 \\ Y = 4 \end{cases}$

5) $\begin{cases} X^2-Y^2=24 \\ X^2+Y^2=36 \end{cases}$

7) $\begin{cases} XY=6 \\ X^2+9Y^2=36 \end{cases}$

2) $\begin{cases} XY=8 \\ -6X+3Y=18 \end{cases}$

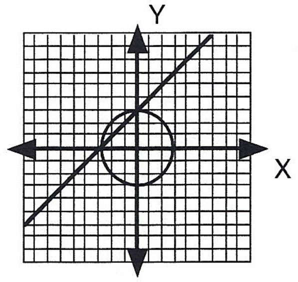
4) $\begin{cases} Y=X+1 \\ 4X^2+Y^2=36 \end{cases}$

6) $\begin{cases} Y=X^2+2 \\ X^2+Y^2=4 \end{cases}$

8) $\begin{cases} Y=2X \\ XY=10 \end{cases}$

Solutions

1) Find the solution of $\begin{cases} Y=X+3 \\ X^2+Y^2=9 \end{cases}$



A) The first equation is a line and the second a circle.

B) Sketch a graph of each to give an estimate of the solutions.

C) Substitute the first into the second to solve for one variable.

$$X^2 + (X+3)^2 = 9$$
$$X^2 + X^2 + 6X + 9 = 9$$
$$2X^2 + 6X = 0$$

D) Factor to find the solutions.

$$X^2 + 3X = 0$$
$$X(X+3) = 0$$
$$X=0 \quad X=-3$$

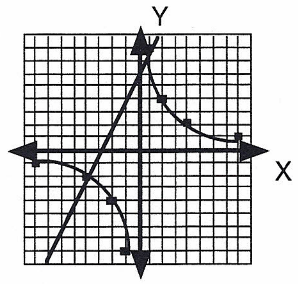
E) Put X=0, and X=-3 into Y = X+3 to find Y.

$Y = 0+3$ $\quad$ $Y = -3+3$

$Y = 3$ $\quad$ $Y = 0$

F) The final solutions are (0,3) and (-3,0)

2) Find the solution of $\begin{cases} XY=8 \\ -6X+3Y=18 \end{cases}$ $\quad Y=2X+6$



A) The first equation is a hyperbola and the second a line.

B) Sketch a graph of each to give an estimate of the solutions.

C) Substitute the second into the first to solve for one variable.

$$X(2X+6) = 8$$
$$2X^2 + 6X - 8 = 0$$
$$X^2 + 3X - 4 = 0$$

D) Factor to find the solutions.

$$(X-1)(X+4) = 0$$
$$X=1 \quad X=-4$$

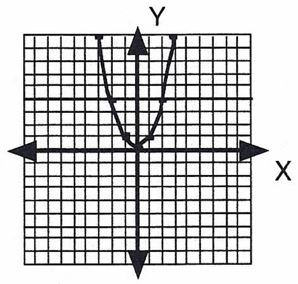
E) Put X=1, and X=-4 into XY = 8 to find Y.

$(1)Y = 8$ $\quad$ $(-4)Y = 8$

$Y = 8$ $\quad$ $Y = -2$

F) The final solutions are (1,8) and (-4,-2)

3) Find the solution of $\begin{cases} Y=X^2 \\ Y=4 \end{cases}$



A) The first equation is a parabola and the second a line

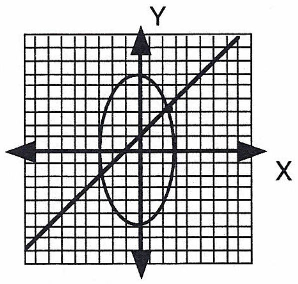
B) Sketch a graph of each to give an estimate of the solutions.

C) Solve for one variable. $\quad 4=X^2$

D) Factor to find the solutions. $\quad \pm 2=X$

E) X=2 & X=-2 and Y=4, so (2,4) and (-2,4)

4) Find the solution of $\begin{cases} 4X^2+Y^2=36 \\ Y=X+1 \end{cases}$



A) The first equation is an ellipse and the second a line.

B) Sketch a graph of each to give an estimate of the solutions.

C) Solve for one variable.

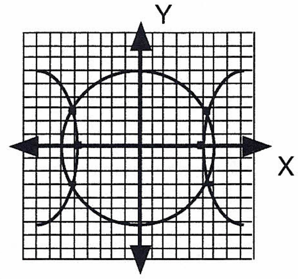
$$4X^2 + (X+1)^2 = 36$$
$$4X^2 + (X^2 + 2X + 1) = 36$$
$$5X^2 + 2X - 35 = 0$$

D) Use the quadratic formula. $\quad X = \dfrac{-1+4\sqrt{11}}{5} \ \& \ \dfrac{-1-4\sqrt{11}}{5}$

E) Y=X+1, so:

$$\left(\frac{-1+4\sqrt{11}}{5}, \frac{4+4\sqrt{11}}{5}\right)\left(\frac{-1-4\sqrt{11}}{5}, \frac{4-4\sqrt{11}}{5}\right)$$

5) Find the solution of $\begin{cases} X^2 - Y^2 = 24 \\ X^2 + Y^2 = 36 \end{cases}$



A) The first equation is a hyperbola and the second, a circle.

B) Sketch a graph of each to give an estimate of the solutions.

C) Eliminate to solve for one variable. $2X^2 = 60$

$X^2 = 30$

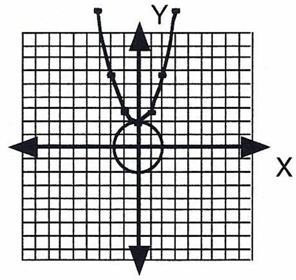
D) Factor to find the solutions. $X = \pm\sqrt{30}$

E) Put $X=\sqrt{30}$, and $X=-\sqrt{30}$ into $X^2+Y^2 = 36$

F) The final solutions are

$(\sqrt{30}, \sqrt{6})(\sqrt{30}, -\sqrt{6})(-\sqrt{30}, \sqrt{6})(-\sqrt{30}, -\sqrt{6})$

6) Find the solution of $\begin{cases} Y = X^2 + 2 \\ X^2 + Y^2 = 4 \end{cases}$



A) The first equation is a parabola and the second, a circle.

B) Sketch a graph of each to give an estimate of the solutions.

C) Substitute the first into the second to solve for one variable.

$(Y-2) + Y^2 = 4$

$Y^2 + Y - 6 = 0$

D) Factor to find the solutions.

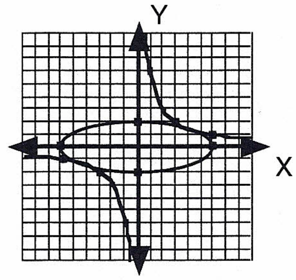
$(Y+3)(Y-2)=0$

$Y=-3^*$ $Y=2$

E) The final solution is (0,2)

*Sometimes when working with quadratics, all the roots do not work. It is a good idea to check the roots to see if they work. In this case Y=-3 does not work in either equation. So we only have one root, which is 2.

7) Find the solution of $\begin{cases} XY = 6 \\ X^2 + 9Y^2 = 36 \end{cases}$



A) The first equation is a hyperbola and the second, an ellipse.

B) Sketch a graph of each to give an estimate of the solutions.

C) Substitute the first into the second to solve for one variable.

$\left(\frac{6}{Y}\right)^2 + 9Y^2 = 36$

$\frac{36}{Y^2} + 9Y^2 = 36$

$36 + 9Y^4 = 36Y^2$

D) Factor to find one variable.

$9Y^4 - 36Y^2 + 36 = 0$

$Y^4 - 4Y^2 + 4 = 0$

$(Y^2 - 2)^2 = 0$

$Y = \pm\sqrt{2}$

E) Put $Y = \pm\sqrt{2}$ into $XY = 6$ to find X.

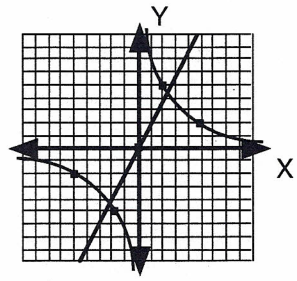
$X(+\sqrt{2}) = 6, X = 3\sqrt{2}$

$X(-\sqrt{2}) = 6, X = -3\sqrt{2}$

F) The final solutions are

$(+3\sqrt{2}, +\sqrt{2})$ & $(-3\sqrt{2}, -\sqrt{2})$

8) Find the solution of $\begin{cases} Y = 2X \\ XY = 10 \end{cases}$



A) The first equation is a line and the second, a hyperbola.

B) Sketch a graph of each to give an estimate of the solutions.

C) Substitute the first into the second to solve for one variable.

$X(2X) = 10$

$2X^2 = 10$

$X^2 = 5$

D) Factor to find the solutions. $X = \pm\sqrt{5}$

E) Put $X=\sqrt{5}$ & $X=-\sqrt{5}$ into $Y = 2X$, to find Y.

F) The solutions are: $(\sqrt{5}, 2\sqrt{5})$ and $(-\sqrt{5}, -2\sqrt{5})$