Lesson 10 Binomial Theorem

So far we've seen the pattern for the coefficients in the triangle. Now take a look at the variables and the exponents. The first study will be the variables. In $(A+B)^2$ the first term in the product has an A but no B. The last term has a B but no A. The middle term has one of each. It will be easier to observe the pattern if we put one of each in all the terms. We can do this by adding a zero exponent in the first and last terms. Notice also that the number of terms is one more than the exponent.

$$1A^2 + 2A^1B^1 + 1B^2 = 1A^2B^0 + 2A^1B^1 + 1A^0B^2$$

Notice how the exponents on A begin with the same exponent to which we are raising the binomial, 2. Then they decrease by 1. For the B's, they begin at 0 and increase by 1 until they get to 2. And if you add the exponents in each term, 2+0, 1+1, 0+2, they all add up to 2. With this knowledge under our belt and the triangle of Pascal, we can predict the product of a binomial raised to a power.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Example 1 Expand $(A + B)^5$.

From the triangle we get the coefficients: 1, 5, 10, 10, 5, 1. Since we are raising to the 5 power we start with A^5B^0 and proceed from there.

$$1A^5B^0 + 5A^4B^1 + 10A^3B^2 + 10A^2B^3 + 5A^1B^4 + 1A^0B^5$$

Example 2 Expand $(A + B)^7$.

From the triangle we get the coefficients: 1, 7, 21, 35, 35, 21, 7, 1. Since we are raising to the 7 power we start with A^7B^0 and proceed from there.

$$1A^7B^0 + 7A^6B^1 + 21A^5B^2 + 35A^4B^3 + 35A^3B^4 + 21A^2B^5 + 7A^1B^6 + 1A^0B^7$$

Practice Problems Tell how many terms there will be and expand.

1) $(A + B)^6$ 2) $(A + B)^4$ 3) $(X + 2)^5$

Solutions

1) 7 terms (6+1) $1A^6B^0 + 6A^5B^1 + 15A^4B^2 + 20A^3B^3 + 15A^2B^4 + 6A^1B^5 + 1A^0B^6$

2) 5 terms (4+1) $1A^4B^0 + 4A^3B^1 + 6A^2B^2 + 4A^1B^3 + 1A^0B^4$

3) 6 terms (5+1) $1X^52^0 + 5X^42^1 + 10X^32^2 + 10X^22^3 + 5X^12^4 + 1X^02^5 = X^5 + 10X^4 + 40X^3 + 80X^2 + 80X + 32$

The Binomial Theorem is a way of predicting what the product of a binomial raised to any power will be without the use of Pascal's Triangle. The key will be how to express the formula in algebraic terms. The triangle can be expressed using factors and fractions. Here are four rows of the triangle. Predict the fifth and sixth rows before turning the page to check.

$$1$$

$$1 \quad \frac{1}{1}$$

$$1 \quad \frac{2}{1} \quad \frac{2\cdot1}{1\cdot2}$$

$$1 \quad \frac{3}{1} \quad \frac{3\cdot2}{1\cdot2} \quad \frac{3\cdot2\cdot1}{1\cdot2\cdot3}$$

Notice how the fractions reduce to come up with the same numbers that we had in the triangle.

$$1 \quad \frac{4}{1} \quad \frac{4\cdot 3}{1\cdot 2} \quad \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3} \quad \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}$$

$$1 \quad \frac{5}{1} \quad \frac{5\cdot 4}{1\cdot 2} \quad \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} \quad \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4} \quad \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$

Now for the algebra to predict the variables. This looks intimidating, but it is because of the increasing and decreasing of the exponents. In the theorem N represents any positive integer, or positive whole number. Here is the Binomial Theorem.

$$(A+B)^N = \underbrace{A^N B^0}_{\text{First Term}} + \underbrace{\frac{N}{1}A^{N-1}B^1}_{\text{Second Term}} + \underbrace{\frac{N(N-1)}{1\cdot 2}A^{N-2}B^2}_{\text{Third Term}} + \underbrace{\frac{N(N-1)(N-2)}{1\cdot 2\cdot 3}A^{N-3}B^3}_{\text{Fourth Term}} \ldots \underbrace{A^0B^N}_{\text{Last Term}}$$

What makes this useful is the ability to know what a specific term will be without expanding the whole product. Look at the third term. There are 2 factors in the coefficient. In the fourth term there are 3. So there is always one less factor in the coefficient than the number of the term. If you wanted to know the 7th term there would be 6 factors. You will notice that the exponent for B is 2 which is one less than the number of the term as well. The exponent for A when added to the exponent for B will always add up to N. Let's do some examples.

Example 3 Find the fourth term of $(X+Y)^6$.

The coefficient will have 3 factors (4 - 1). $\frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3} = 20$

The exponent of Y will be 3, (4 - 1), and the exponent for X will also be 3 since 3 + 3 = 6. X^3Y^3

Putting it all together: $20X^3Y^3$

Example 4 Find the third term of $(X+2)^5$.

The coefficient will have 2 factors (3 - 1). $\frac{5\cdot 4}{1\cdot 2} = 10$

The exponent of 2 will be 2, (3 - 1), and the exponent for X will also be 3 since 2 + 3 = 5. $X^3 2^2$

Putting it all together: $10X^3 2^2 = 10X^3(4) = 40X^3$

Practice Problems Tell how many terms and expand.

1) $(A+3)^5$ 2) $(3X+4)^3$ 3) $(X-Y)^4$

4) What is the 3rd term in $(P+Q)^6$? Hint: $X - Y = X + (-Y)$

5) What is the 4th term in $(3A+B)^5$?

6) What is the 3rd term in $(2C-D)^4$?

Solutions

1) 6 terms (5+1) $1A^5 3^0 + 5A^4 3^1 + 10A^3 3^2 + 10A^2 3^3 + 5A^1 3^4 + 1A^0 3^5 = A^5 + 15A^4 + 90A^3 + 270A^2 + 405A + 243$

2) 4 terms (3+1) $1(3X)^3 4^0 + 3(3X)^2 4^1 + 3(3X)^1 4^2 + 1(3X)^0 4^3 = 27X^3 + 108X^2 + 144X + 64$

3) 5 terms (4+1) $1X^4(-Y)^0 + 4X^3(-Y)^1 + 6X^2(-Y)^2 + 4X^1(-Y)^3 + 1X^0(-Y)^4 = X^4 - 4X^3Y + 6X^2Y^2 - 4XY^3 + Y^4$

4) What is the 3rd term in $(P+Q)^6$? $\frac{6\cdot 5}{1\cdot 2} P^4Q^2 = 15P^4Q^2$

5) What is the 4th term in $(3A+B)^5$? $\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} (3A)^2B^3 = 90A^2B^3$

6) What is the 3rd term in $(2C-D)^4$? $\frac{4\cdot 3}{1\cdot 2} (2C)^2(-D)^2 = 24C^2D^2$