Lesson 25 Parabola, Maxima and Minima

So far any parabola that we have graphed has moved up and down the Y axis. If it has a positive coefficient as in $Y = 2X^2$, then the vertex, or lowest point, is on the Y axis. If it has a negative coefficient as in $Y = -2X^2$, then its highest point, or vertex, is on the Y axis. (For equations of the sort $X = Y^2$, with both positive and negative coefficients, this holds true with the X-axis). But if the quadratic has a middle term, then the parabola moves off the axis. Where it moves, and how to predict where the vertex will be located, is the object of this lesson. The vertex, or lowest point, of a positive parabola is called the minima. The highest point, or vertex, of a negative parabola, is called the maxima.

Example 1 Graph $Y = X^2 + 4X - 2$ by plotting several points.

X=-5 $Y = (-5)^2 + 4(-5) - 2$
$Y = +3$

X=-4 $Y = (-4)^2 + 4(-4) - 2$
$Y = -2$

X=-3 $Y = (-3)^2 + 4(-3) - 2$
$Y = -5$

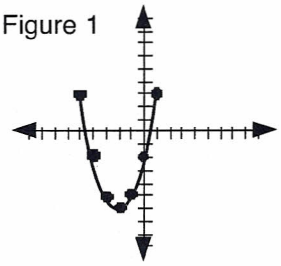
X=-2 $Y = (-2)^2 + 4(-2) - 2$
$Y = -6$

X=-1 $Y = (-1)^2 + 4(-1) - 2$
$Y = -5$

X=0 $Y = (0)^2 + 4(0) - 2$
$Y = -2$

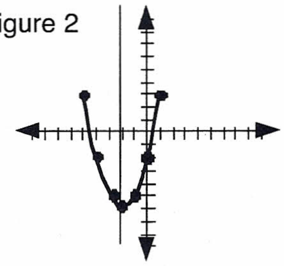
X	Y
-5	+3
-4	-2
-3	-5
-2	-6
-1	-5
0	-2
1	+3

Figure 1



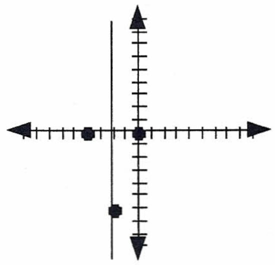
The graph shows visually what our table of data was telling us. The pattern is that it decreases to -6 then begins moving up again. So the vertex, or minima, is (-2, -6). But how could we derive this from the original equation? We know that the -2, or (remembering $AX^2 + BX + C$) the "C" term, moves the parabola up or down the Y axis. So we need to focus on the middle term, with the "B" coefficient. In the example, we won't worry about the C term but instead focus on $Y = X^2 + 4X$. Now looking at the graph, the parabola shifted to the left instead of staying on the Y axis. I drew a line through the vertex which splits the parabola into two symmetrical pieces. This line, parallel to the Y axis, is called the axis of symmetry.

Figure 2



On the graph, we can see that the X coordinate of this line is -2. Plugging -2 into the equation gives us Y=-6, which is our vertex. So, if we can find the value of the X coordinate, then we can find the Y coordinate, and we know the location of the vertex. Let's set Y=0 to find the value of X.

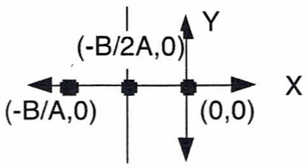
$Y = X^2 + 4X$
$0 = X^2 + 4X$
$0 = X(X+4)$
$X=0$ or $X+4=0$
$X=0$ $X=-4$



We can see that the axis of symmetry, which is X=-2, will lie half way between these points, and we know if X=-2, then Y=-6.

Now let's run through the same process with $AX^2 + BX + C = 0$, focusing on $AX^2 + BX$.

$Y = AX^2 + BX$
$0 = AX^2 + 4X$
$0 = X(AX+B)$
$X=0$ or $AX+B=0$
$X=0$ $X=-B/A$



If the two coordinates are 0 and -B/A, then the distance halfway between them to find the line of symmetry, according to the midpoint formula, is -B/2A or 1/2 times -B/A.

So, when given an equation with B (a middle term), use -B/2A to find the axis of symmetry (X coordinate), then use this in the original equation to find the value of the Y coordinate, and then you have the vertex.

Example 2 Graph $Y = X^2 + 5X + 2$ $\frac{-B}{2A} = \frac{-(5)}{2(1)} = -5/2$

A=1, B=5, C=2

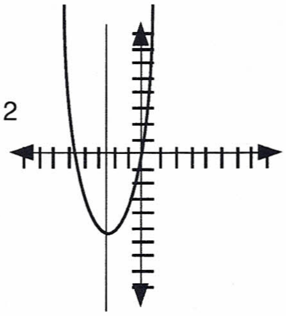
$Y = X^2 + 5X + 2$

Plugging in the x-coordinate of the axis of symmetry:

$Y = (-5/2)^2 + 5(-5/2) + 2 = -17/4$ or $-4\ 1/4$

Vertex = (-2 1/2, -4 1/4)

Figure 2



Practice Problems *Find the axis of symmetry and the vertex then sketch the graph.*

1) $Y = X^2 - 6X + 1$
2) $Y = 2X^2 + 8X + 2$
3) $Y = -X^2 + 3$
4) $Y = -3X^2 + 6X$
5) $Y = 2/3X^2 - 4$
6) $Y = 1/2X^2 - 2X$

Solutions

1) $Y = X^2 - 6X + 1$

A=1, B=-6

$\frac{-B}{2A} = \frac{-(-6)}{2(1)} = 3$

$Y = X^2 - 6X + 1$

$Y = (3)^2 - 6(3) + 1 = -8$

Axis of Symmetry @ X=3

Vertex = (3,-8)

2) $Y = 2X^2 + 8X + 2$

A=2, B=8

$\frac{-B}{2A} = \frac{-(8)}{2(2)} = -2$

$Y = 2X^2 + 8X + 2$

$Y = 2(-2)^2 + 8(-2) + 2 = -6$

Axis of Sym. @ X=-2

Vertex = (-2,-6)

3. Points down, vertex at (0,3)

4) $Y = -3X^2 + 6X$

A=-3, B=+6

$\frac{-B}{2A} = \frac{-(6)}{2(-3)} = 1$

$Y = -3X^2 + 6X$

$Y = -3(1)^2 + 6(1) = 3$

Axis of Sym.@ X=1

Vertex = (1,3)

5. Points up, vertex at (0,-4)

6) $Y = 1/2X^2 - 2X$

A=1/2, B=-2

$\frac{-B}{2A} = \frac{-(-2)}{2(1/2)} = 2$

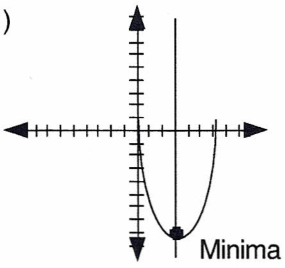
$Y = 1/2X^2 - 2X$

$Y = 1/2(2)^2 - 2(2) = -2$

Axis of Sym. @ X=2

Vertex = (2,-2)

1)



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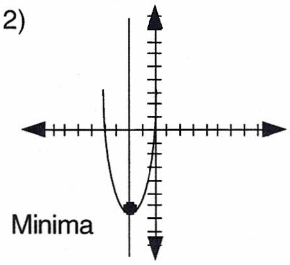
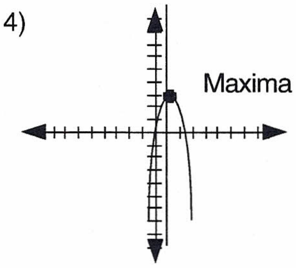
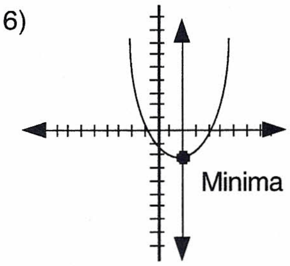


4)



6)



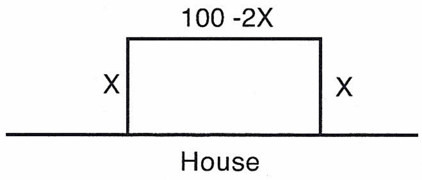
Summary of the Parabola in Algebrese

For any quadratic of the form: $Y = AX^2 + BX + C$

1) If $A > 0$, the graph of the parabola points up
2) If $A < 0$, the graph of the parabola points down
3) If $|A| > 1$, the graph is steeper than $Y=X^2$
4) If $0 < |A| < 1$, the graph is flatter than $Y=X^2$
5) C moves the graph up & down the Y axis.
6) The axis of symmetry is $X = -B/2A$
7) The coordinates of the vertex are $[-B/2A, A(-B/2A)^2 + B(-B/2A) + C]$
8) The vertex is the maxima or the minima.

Now that we know how to find the maxima and the minima, we can apply this knowledge to solve some real life problems.

Example 3 You purchased 100 feet of fence to come off the back of your house for a play area. What will the dimensions be to give you the maximum area for your yard?



The area is the length times the width, or (X)(100-2X).

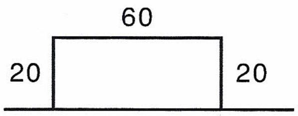
$(X)(100-2X) = 100X - 2X^2 = -2X^2 + 100X$ Points down

A=-2, B=+100 $\frac{-(100)}{2(-2)} = 25$ is the axis of symmetry

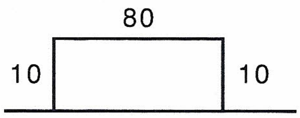
Area = $-2(25)^2 + 100(25) = 1250$ (25, 1250) is the vertex

Area = 25 x 50 = 1250 sq ft

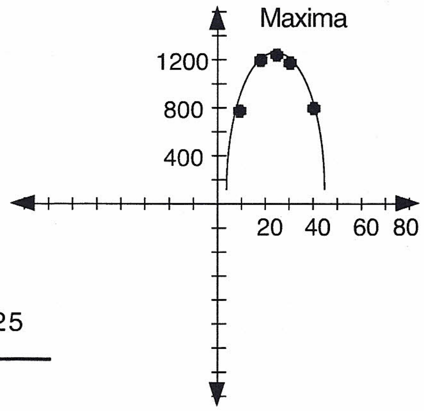
Notice the different combinations and their place on the graph, and see if the solution makes sense.

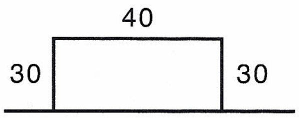


If X=20, then the area is 1,200

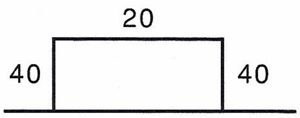


If X=10, then the area is 800

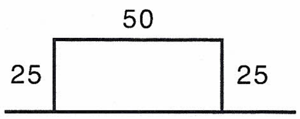




If X=30, then the area is 1,200



If X=40, then the area is 800



If X=25, then the area is 1,250

Practice Problems

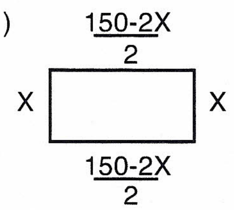
1) Having just bought some chickens, you need to fence in a rectangular chicken yard. If you have 150 feet of fencing, what will the dimensions of the largest yard you can make be?

2) The weather man is calling for a frost. It is up to you to cover the tender shoots tonight. In the barn there is a roll of sheet metal, 24 inches wide, that will, when folded twice, make a cover for the plants. What is the height and breadth of the rectangular dimensions that will give the most space underneath?

3) Chuck has a 20 foot piece of wood to make a sandbox. What will the most efficient use of the lumber to get maximum space with this amount of timber?

Solutions

1)



Area = (X)(150-2X)/2 = (X)(75-X)

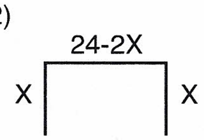
$(X)(75-X) = 75X - X^2 = -X^2 + 75X$ Points down

A=-1, B=+75 $\frac{-(75)}{2(-1)} = 37.5$ is the axis of sym.

Area = $-(37.5)^2 + 75(37.5) = -1406.25 + 2812.5$
= 1406.25 so (37.5, 1406.25) is the vertex

Area = 37.5 x 37.5 = 1406.25 sq ft

2)



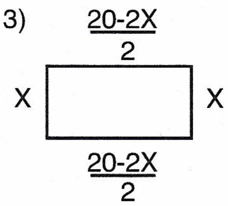
Area = $(X)(24-2X) = 24X-2X^2 = -2X^2+24X$
Points down

A=-2, B=+24 $\frac{-(24)}{2(-2)} = 6$ is the axis of sym.

Area = $-2(6)^2 + 24(6) = -72 + 144 = 72$
(6, 72) is the vertex

Area = 6 x 12 = 72 sq in

3)



Area = $(X)(20-2X)/2 = 10X-X^2 = -X^2+10X$
Points down

A=-1, B=+10 $\frac{-(10)}{2(-1)} = 5$ is the axis of sym.

Area = $-(5)^2 + 10(5) = -25+50 = 25$
(5,25) is the vertex

Area = 5 x 5 = 25 sq ft