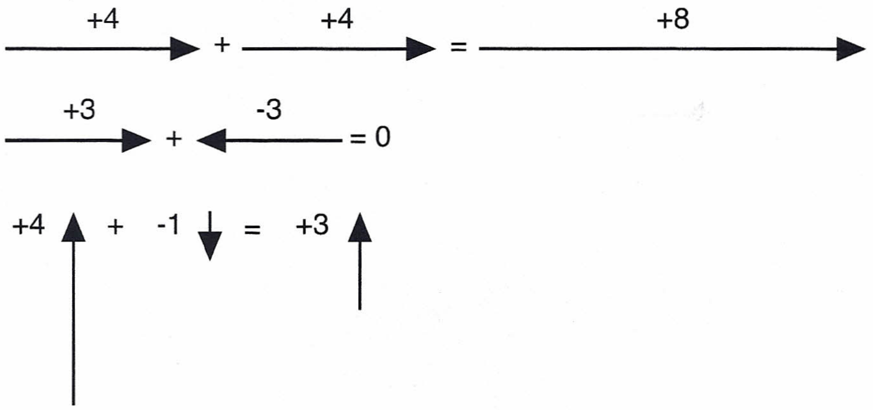
Lesson 31 Vectors

Vectors have two components: direction and magnitude. They are shown graphically as arrows. Motions in one dimension (along a line) give their direction in terms of a sign. (+ or -) We have been studying a simple form of one-dimensional vectors since we began studying positive and negative numbers in Pre-Algebra, without using the same name. Consider the following examples:



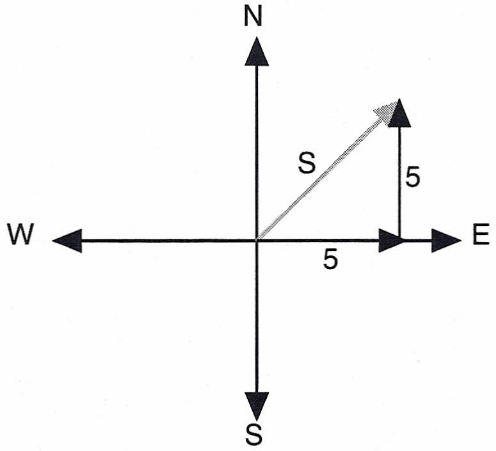
$(+4) + (+4) = (+8)$

$(+3) + (-3) = (0)$

$(+4) + (-1) = (+3)$

Vectors move in two dimensions, over and up, instead of one direction as in a number line. This movement or motion takes place in a two dimensional plane.

Example 1



You leave your hotel to solve a national emergency and head 5 blocks east, then turn at a right angle and head 5 blocks north. When you arrive there you call Superman. Since he doesn't need to go "on foot", he wants to know how far you are and in what direction. The length of the distance between the hotel and where you now are is called the magnitude of the vector. On the picture, "S" stands for Superman. The direction is called simply that. Every vector has these 2 components, magnitude and direction.

Since we have a right triangle, we can figure out the length of S by using the Pythagorean Theorem. $S = \sqrt{50}$ or $5\sqrt{2}$, which our calculator shows us is 7.07 blocks.

We have computed the distance, and the direction is north and east, or northeast. So we tell the man in red and blue, head northeast for 7.07 blocks.

$5^2 + 5^2 = 50 = S^2$

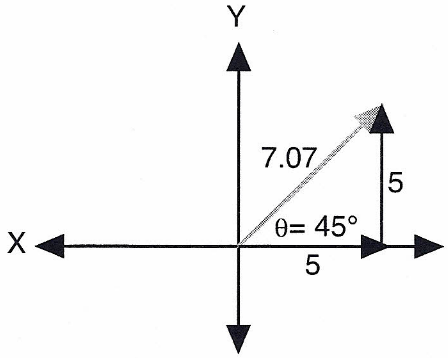
$\sqrt{50} = S$

7.07 blocks = S, which is the magnitude of the resultant vector

Most of our vector problems will not be drawn as a grid of downtown, but with a grid we are familiar with, the Cartesian coordinate system, which consists of 2 number lines. Instead of north and south, we have the Y axis. We refer to the X axis in place of east and west.

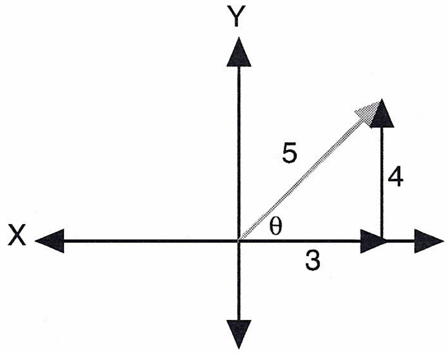
In our first example, we did know that the direction of the resultant vector was 45° because of what we learned in *Geometry* about special right triangles. If the 2 legs of a right triangle are the same length, then the triangle is a 45°-45°-90° right triangle. We could have told the Man of Steel to travel 45° for 7.07 blocks. The direction is referred to as angle theta, using the Greek letter θ.

Example 1



Suppose you begin at your home and travel 3 miles due east and then travel an additional 4 miles due north. In order to compute the magnitude of the journey from start to finish, we use the Pythagorean Theorem. We are now adding two 1-dimensional vectors that are at right angles to each other. The answer is called the magnitude of the resultant vector, or the resultant.

Example 2



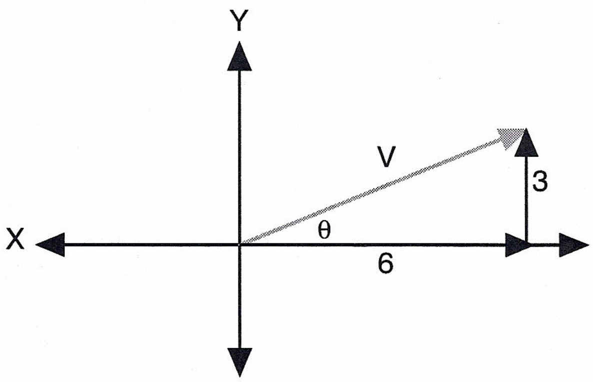
$3^2 + 4^2 = 25 =$ resultant^2 or magnitude^2

magnitude = 5 miles

Theta appears to be a little larger than 45°, so we estimate that it is about 50° to 55°. We'll find out how to determine the measure of angle theta later on. This process of discovering the angle using the length of the sides is taught more fully in trigonometry, which you will study more fully in *PreCalculus*.

Find the magnitude of a vector which is the result of 2 vectors, one moving along the X axis 6 blocks, and the other added to the first one moving 3 blocks in the direction of Y.

Example 3

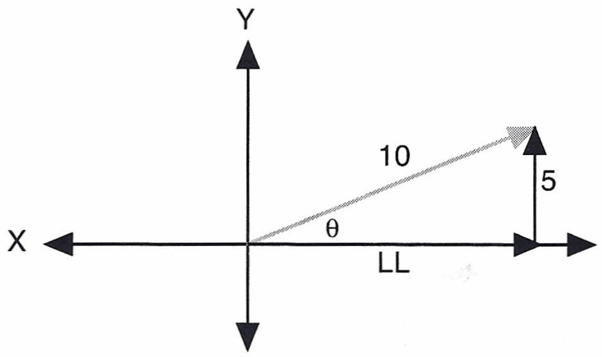


$3^2 + 6^2 = 45 = V^2$

$\sqrt{45} = V \approx 6.7$

If the magnitude of the vector is 10 and the short leg is 5, what is the length of the long leg and what is the measure of theta?

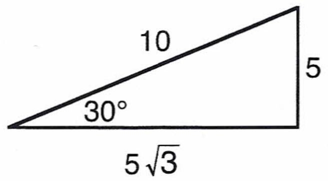
Example 4



$$5^2 + LL^2 = 10^2$$

$$LL = \sqrt{75}$$

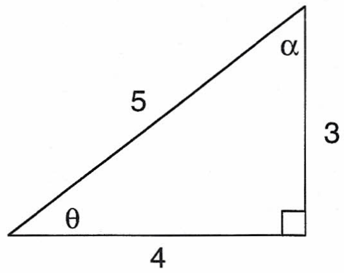
$$LL = 5\sqrt{3}$$



Using geometry skills, we can see that the length of the X vector is $5\sqrt{3}$. Using what we know of special right triangles, we can see this is a 30°-60°-90° triangle because the short side, or short leg, is 1/2 the length of the hypotenuse, which is our resultant vector. This ratio of 1 to 2, or 1/2, or .5, is the door to our understanding trigonometry, which helps us find theta, or direction. Now for a brief review of trigonometry as presented in the Math-U-See *Geometry* book.

Let's begin by describing all the angles and all the sides of a right triangle. We already know one angle is 90°, and the side opposite the right angle is the hypotenuse. This leaves two angles and two sides, or legs, to name. I've decided to call the angles θ (theta) and α (alpha), both letters in the Greek alphabet. The sides are described in reference to the angles. If I am standing in angle θ, then the leg furthest away from me will be the "opposite" side. The side, or leg, that touches me (where my feet are standing in the illustration) is the "adjacent" side. If I move to stand in angle α, then the side furthest away from me becomes the "opposite" side and the side touching me is the "adjacent" side. The names for the sides depend on what angle you are referring to.

I'll illustrate this with our old friend, the 3-4-5 right triangle.



Standing at θ, the opposite side is 3 units long and the adjacent side is 4 units long. If I move to α, then 4 is "opposite" and 3 is "adjacent". In both instances, the hypotenuse is 5.

Trigonometric Ratios Now we come to the main three trigonometric ratios, using the terminology of opposite, adjacent, and hypotenuse. They are sine, cosine, and tangent. The sine of either angle in a right triangle, in our example θ or α, is described as the ratio of the opposite side over the hypotenuse.

A fun way to remember these three trigonometric ratios is the result of dropping a brick on your big toe. What would you do? Probably get a pan of water and "soak your toe", or SOH-CAH-TOA.

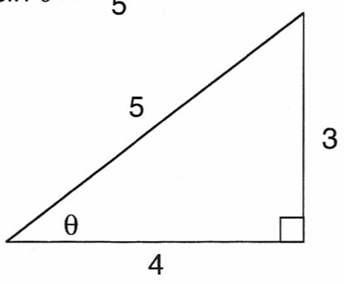
SOH stands for $\sin = \dfrac{\text{opposite}}{\text{hypotenuse}} \Rightarrow S = \dfrac{O}{H}$

CAH stands for $\cos = \dfrac{\text{adjacent}}{\text{hypotenuse}} \Rightarrow C = \dfrac{A}{H}$

TOA stands for $\tan = \dfrac{\text{opposite}}{\text{adjacent}} \Rightarrow T = \dfrac{O}{A}$

Notice the commonly used abbreviations for sine, cosine, and tangent.

Using the 3-4-5 right triangle, the sine of angle θ is $\dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{3}{5}$ or $\sin\theta = \dfrac{3}{5}$



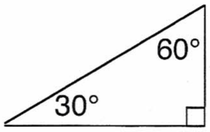
The cosine of θ is the adjacent over the hypotenuse, or $\cos\theta = \dfrac{4}{5}$

And the tangent of θ is the opposite over the adjacent, or $\tan\theta = \dfrac{3}{4}$

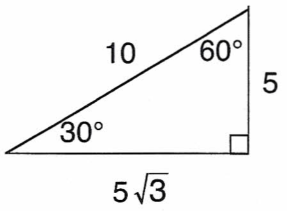
Another relationship found in these trigonometric expressions is that sine and cosine are complementary angles. In a right triangle (with one right angle) the other two angles always add up to 90°, so they are complementary.

Using the 3-4-5 triangles, let's look at the ratios once again. We learned that the ratios are constant for a 45°-45°-90° triangle or a 30°-60°-90° triangle. In the latter, the short side is always 1/2 the hypotenuse. The length of the sides of all the 30°-60°-90° right triangles may vary, but the short side will always be 1/2 of the hypotenuse.

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{short side}}{\text{hypotenuse}} = \frac{1}{2}$$

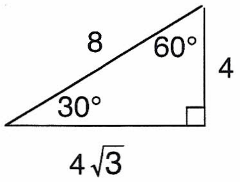


If the sides of the triangle had lengths such as:



then the sin 30° $= \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{5}{10} = \dfrac{1}{2}$

or if the side lengths varied and you had:



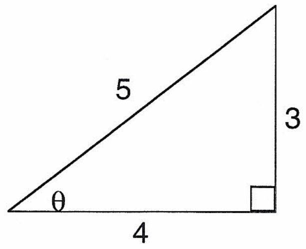
then sin 30° $= \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{4}{8} = \dfrac{1}{2}$

Observe that the ratio of the small side to the hypotenuse remains constant.

If you type in 30° on your calculator and then hit the key that says SIN, it will give you the ratio as .5, because the ratio is always one half or .5. Trig ratios are usually expressed as a decimal with 4 digits after the decimal point, so .5 would be written as .5000. If you have the angle, you can always find the ratio. The inverse is also true. In our examples, we have the length of the sides but not the angle. We are hoping that if we have the ratio, we can go in reverse and find the angle, and this is true. Calculators differ, so you may have to get the manual out to find out how to compute trig functions on your model. Use the sin 30° = .5 as your standard to figure out how to use your calculator.

To find the measure of the angle, first make sure that your calculator gives the answer in degrees and not radians or some other kind of measure. The inverse of X is 1/X or X^{-1} power. So also in trigonometry, the inverse of sin is $\sin^{-1}$. There is usually a second line or inverse key on your calculator. It is similar to a shift key, and is used to get the inverse of the sin, cos, and tan.

If you were to type in .5000 on your calculator and then hit the key that says SIN^{-1}, it will give you the answer as 30°. Given the ratio, you can always find the angle. In the next example find the angles using the ratios of the 3-4-5 right triangle.



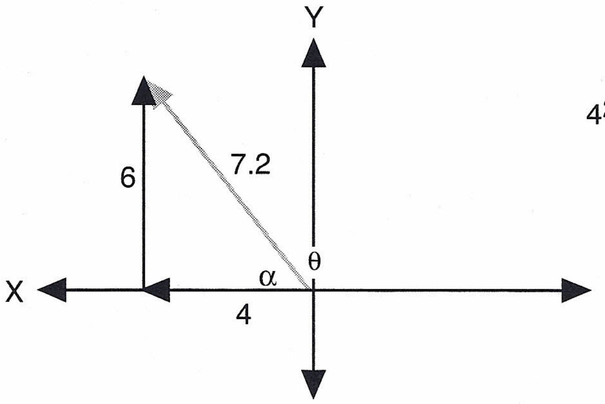
$\sin\theta = \frac{3}{5} = .6000$ enter .6000, hit SIN^{-1} $\theta = 36.9°$

$\cos\theta = \frac{4}{5} = .8000$ enter .8000, hit COS^{-1} $\theta = 36.9°$

$\tan\theta = \frac{3}{4} = .7500$ enter .7500, hit TAN^{-1} $\theta = 36.9°$

In the next problem, find the magnitude and direction of the resultant vector.

Example 5



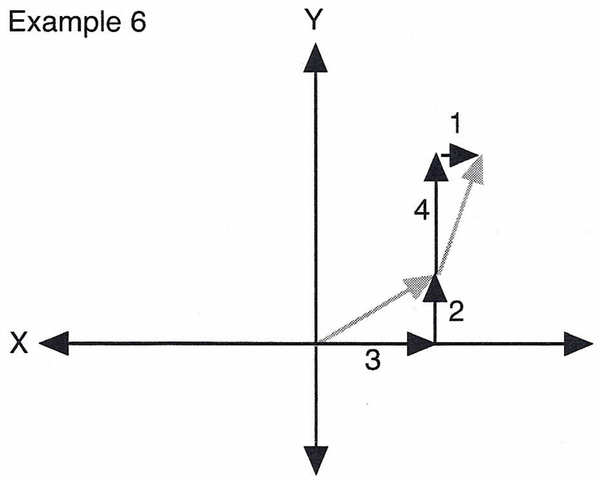
$$4^2 + 6^2 = 52 = V^2$$

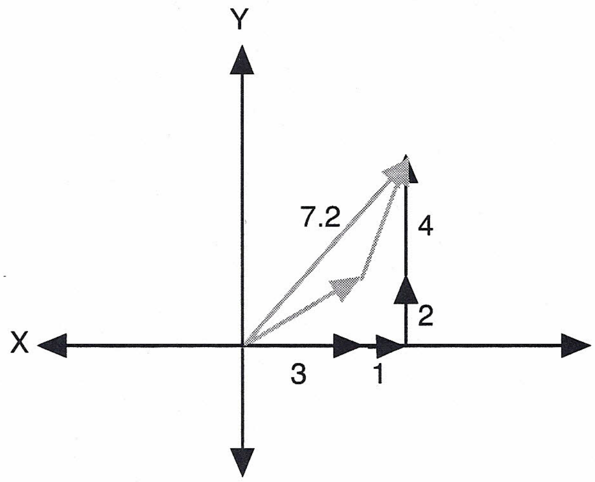
$$\sqrt{52} = V \approx 7.2$$

The direction, we can see, is more than 90°. If we find the angle inside the triangle, which we'll call α, we can subtract from 180° to find the measure of θ since $\alpha + \theta = 180°$. Using the tangent, we see the side opposite over the side adjacent is 6/4 or 1.5. Entering 1.5 and then pushing the key for TAN^{-1} we see the angle is close to 56° inside, so angle θ is 180° - 56° or 124°. The magnitude is 7.2 and the direction is 124°.

If you are adding multiple vectors, make sure you draw them with the tail of the second vector placed over the tip or head of the first vector. See Example 6. The first vector is over 3 and up 2. The second vector, added to it, is over 1 and up 4. The sum of these 2 vectors is shown in the second figure.

Example 6





You can find this vector's over and up coordinates by adding the X, or over, dimensions, (3 + 1 = 4), then adding the Y dimensions (2 + 4 = 6). The final vector is over 4 and up 6, which is what we had in the previous example, so we know the magnitude is 7.2 and the direction is 56°.