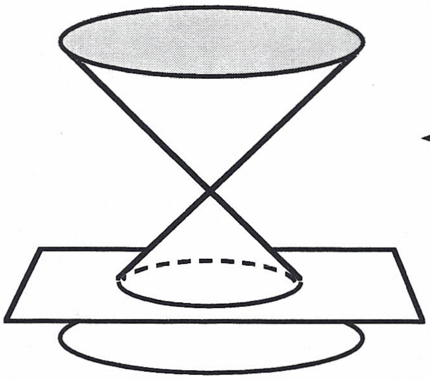
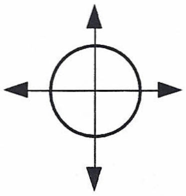
Lesson 23 Conic Sections, Circle and Ellipse

Conic sections were introduced in Algebra 1 as the circle, ellipse, parabola, and hyperbola. In this book we want to cover them in more depth both as they appear on a two dimensional graph, and their corresponding equations. These four shapes are also referred to as conic sections. Conic comes from cone. When a plane intersects a single or double cone the resultant shapes are: the circle, ellipse, parabola, and hyperbola. Look at the shapes below and on the video.

Circle

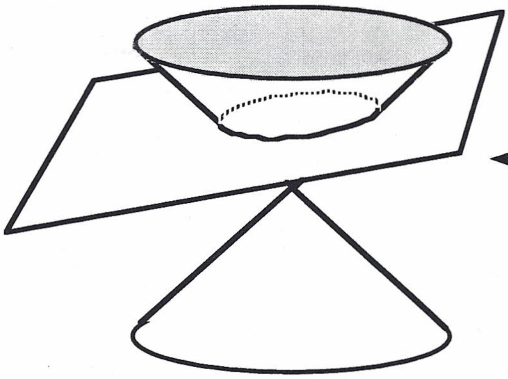


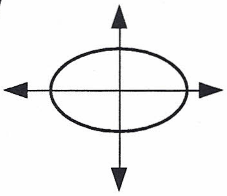


$X^2 + Y^2 = 36$

$AX^2 + AY^2 = R^2$

Ellipse

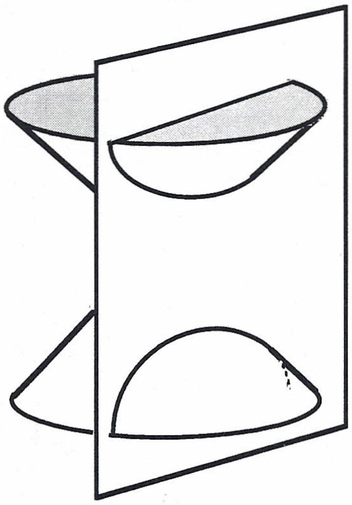


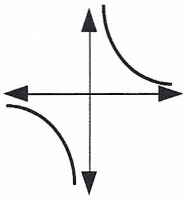


$4X^2 + 9Y^2 = 36$

$AX^2 + BY^2 = R^2$

Hyperbola

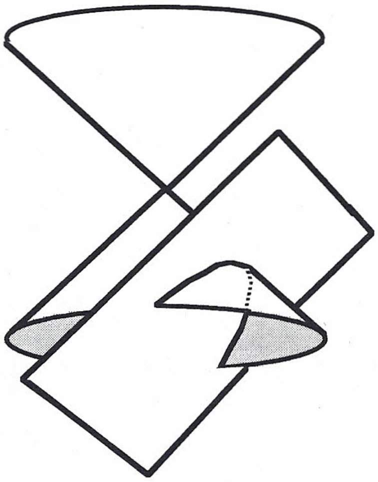


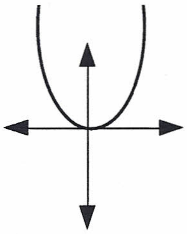


$XY = 6$

$X^2 - 2Y^2 = C^2$

Parabola





$Y = 2X^2$

$Y = AX^2$

The Circle The equation of the circle could better be written as $(X-a)^2 + (Y-b)^2 = R^2$ with the center at (a,b) and a radius of R. $X^2 + Y^2 = 9$ can be rewritten in this form as $(X-0)^2 + (Y-0)^2 = 3^2$ with the center at (0,0) and a radius of 3. Its graph is Figure A.

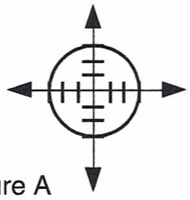


Figure A

Example 1

Graph $X^2 + Y^2 = 25$

The equation of the circle could better be written as $(X-0)^2 + (Y-0)^2 = 5^2$ with the center at (0,0) and a radius of 5. Its graph is Figure 1.

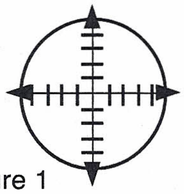


Figure 1

Example 2

Sometimes we get a quadratic for X and Y such as: $X^2 + 2X + 1 + Y^2 - 4Y + 4 = 9$.

Factoring $(X+1)^2 + (Y-2)^2 = 3^2$

The center is (-1, 2) and the radius is 3. The graph is Figure 2.

If X=-1 and Y=2, then you have the same graph as in Figure A with the same radius.

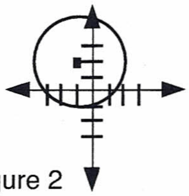


Figure 2

Example 3

Rewrite the equation to find the center and the radius: $X^2 - 6X + Y^2 + 10Y = -18$.

To make the X and Y components perfect squares, we need to complete the squares of each.

$X^2 - 6X + \_\_\_ + Y^2 + 10Y + \_\_\_ = -18 + \_\_\_$

$X^2 - 6X + 9 + Y^2 + 10Y + 25 = -18 + 34$

Factoring $(X-3)^2 + (Y+5)^2 = 4^2$

The center is (3, -5) and the radius is 4. The graph is Figure 3.

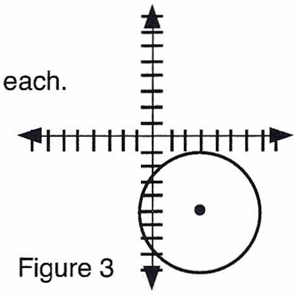


Figure 3

Example 4

Find the equation of the circle given the center point and the radius. The center is (-2, 3) and the radius is 4.

Working backwards, $(X+2)^2 + (Y-3)^2 = 4^2$, which is $X^2 + 4X + 4 + Y^2 - 6Y + 9 = 16$.

Combined further, $X^2 + 4X + Y^2 - 6Y = 3$.

Practice Problems

Find the coordinates of the center and the radius, then graph the result.

1) $X^2 + Y^2 = 16$
2) $(X+1)^2 + (Y+1)^2 = 36$
3) $(X-2)^2 + (Y+3)^2 = 49$
4) $4X^2 + 4Y^2 = 9$

Given the coordinates of the center and the radius, create the equation of the circle.

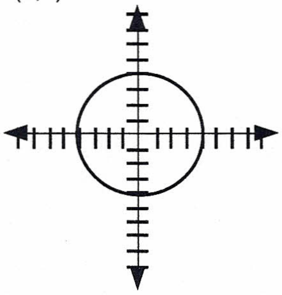
5) (1,2) r = 4
6) (-2,-3) r = 2
7) (0,4) r = 8
8) (3, 1/2) r = 10

By completing the square, find the center and radius of these equations, then sketch the results.

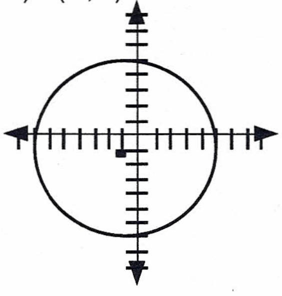
9) $X^2 - 6X + Y^2 - 8Y = -8$.
10) $X^2 - 2X + Y^2 - 4Y = 11$.
11) $X^2 + Y^2 - 8Y - 9 = 0$.
12) $X^2 - X + Y^2 + 2Y = -29/36$

Solutions

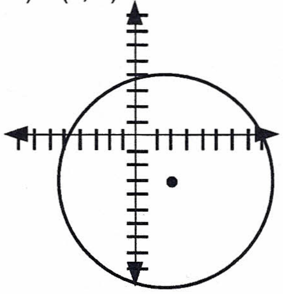
1) (0,0) r = 4



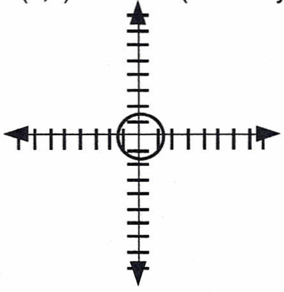
2) (-1,-1) r = 6



3) (2,-3) r = 7



4) (0,0) r = 3/2 (divide by 4)



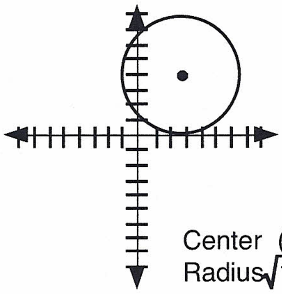
5) $(X-1)^2 + (Y-2)^2 = 16$

6) $(X+2)^2 + (Y+3)^2 = 4$

7) $(X)^2 + (Y-4)^2 = 64$

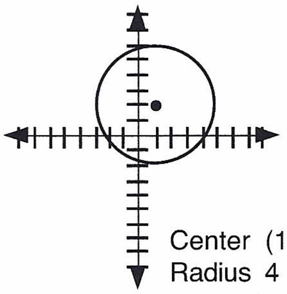
8) $(X-3)^2 + (Y-1/2)^2 = 100$

9) $(X-3)^2 + (Y-4)^2 = 17$



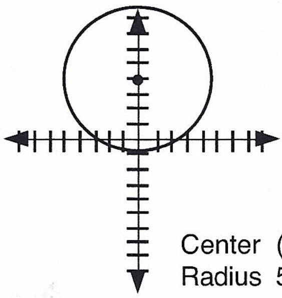
Center (3,4)
Radius $\sqrt{17}$

10) $(X-1)^2 + (Y-2)^2 = 16$



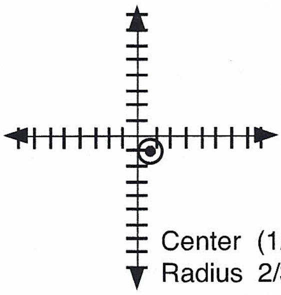
Center (1,2)
Radius 4

11) $(X-0)^2 + (Y-4)^2 = 25$



Center (0,4)
Radius 5

12) $(X-1/2)^2 + (Y+1)^2 = 4/9$



Center (1/2,-1)
Radius 2/3

The Ellipse If the coefficients of X^2 and Y^2 are equal, then you have a circle. In our previous example, both coefficients were 1. If you were given an equation with coefficients of 4, you could divide through the equation by 4 and they would be 1 again. If the coefficients are equal, the graph is a circle. The equation $9X^2 + 4Y^2 = 36$ (or the same equation after dividing by 36, $X^2/4 + Y^2/9 = 1$) looks similar to an equation for a circle, because you have two squares added together. In this case observe the coefficients. If the coefficients are not equal, then the equation is for an ellipse.

Example 1 Plot several points and graph the ellipse $4X^2 + 9Y^2 = 36$.
The key is to find the value of each variable which makes the corresponding term equal 0. Then you know where the graph intercepts the axes.

X	Y
0	±2
±3	0

If $X = 0$:
$$4(0) + 9Y^2 = 36$$
$$9Y^2 = 36$$
$$Y^2 = 4$$
$$Y = \pm 2$$

If $Y = 0$:
$$4X + 9(0)^2 = 36$$
$$4X^2 = 36$$
$$X^2 = 9$$
$$X = \pm 3$$

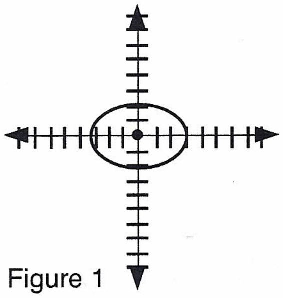


Figure 1

Example 2 Plot several points and graph the ellipse $9(X-1)^2 + 16(Y-2)^2 = 144$.
Locate the center, then find the values of X and Y which make each term equal zero. Then you can find the extremities of the ellipse.

X	Y
1	+5
1	-1
-3	2
+5	2

If $X = 1$:
$$9(0) + 16(Y-2)^2 = 144$$
$$16(Y-2)^2 = 144$$
$$(Y-2)^2 = 9$$
$$Y-2 = +3, -3$$
$$Y = +5, -1$$

If $Y = 2$:
$$9(X-1)^2 + 16(0) = 144$$
$$9(X-1)^2 = 144$$
$$(X-1)^2 = 16$$
$$X-1 = +4, -4$$
$$X = +5, -3$$

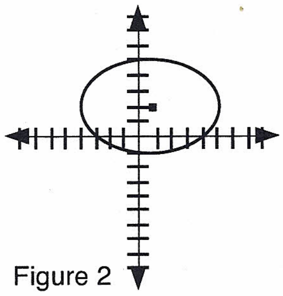


Figure 2

Practice Problems *Find the coordinates of the center and graph the result.*

1) $9X^2 + 4Y^2 = 36$

2) $2(X-1)^2 + 3(Y+1)^2 = 48$

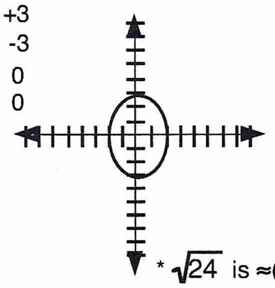
3) $\dfrac{(X+1)^2}{25} + \dfrac{(Y+2)^2}{20} = 1$

4) $16X^2 + 9Y^2 = 144$

Solutions

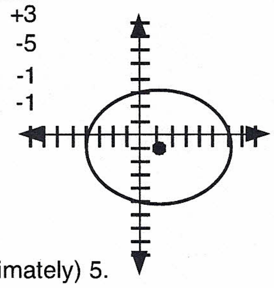
1)

X	Y
0	+3
0	-3
+2	0
-2	0



2)

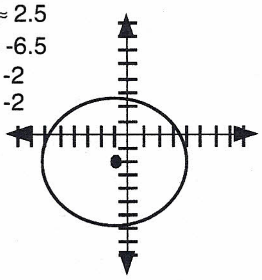
X	Y
1	+3
1	-5
-4*	-1
+6*	-1



* $\sqrt{24}$ is ≈(approximately) 5.

3)

X	Y
-1	≈ 2.5
-1	≈ -6.5
+4	-2
-6	-2



4)

X	Y
0	+4
0	-4
-3	0
+3	0